Worcester County Mathematics League

Varsity Meet 3
January 27, 2016

COACHES' COPY
ROUNDS, ANSWERS, AND SOLUTIONS

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016
Round 1: Similarity and Pythagorean Theorem

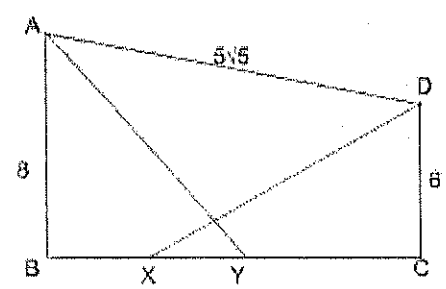
All answers must be in simplest exact form in the answer section

NEITHER CALCULATOR NOR RULER ALLOWED

1. In $\triangle ABC$, AB = 5, BC = 12 and $AB \perp BC$. Find the distance from point B to AC.

2. CD is a line segment, and point B is on line segment CD. Point A and point E are outside the line segment CD such that CE bisects $\angle ACD$ and BE bisects $\angle ABD$. If $\angle BAC = 80^\circ$, find $\angle BEC$ in degrees.

3. $\angle B$ and $\angle C$ are right angles, AB = 8, DC = 6, AD = $5\sqrt{5}$, BX = XY and XD = AY. Find YC.



ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________°

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016
Round 2: Algebra 1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $2x+y=5$, and $y+2z=3$, what is the value of $x-z$?

2. Philip is driving back home. For the first half of the trip ("half" in terms of distance), he drives 40 miles per hour. For the second half of the trip, he drives 60 miles per hour instead. What is the average speed of this trip?

3. Rachel walks on a moving walkway at Logan airport. She walks at a constant speed all the time, both back-and-forth. It takes her 15 seconds to walk to the end of the walkway and then back to the starting point. If the walkway doubles its speed, it takes her 24 seconds to walk to the end and back to the starting point. If the moving walkway is off, how long will it take for Rachel to walk to the end of the walkway and back to the starting point at her constant speed?

ANSWERS

(1 pt.) 1. ______________________________

(2 pts.) 2. ______________________________ miles per hour

(3 pts.) 3. ______________________________ seconds

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016
Round 3: Functions

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $c(x) = x \bmod 12$ is a clock function (for example, $c(0) = 12$, $c(13) = 1$ and $c(-8) = 4$), find the value of $c(\,c(42) + c(-14)\,)$.

2. Abby, Ben and Catherine are collecting candies at Halloween. Afterwards, Abby puts all candies into a pile and gives half plus three pieces of the total candy to Ben. Then, Ben gives a third plus two pieces of his candy to Catherine. If Catherine receives n (Assume $n>3$) pieces of candy from Ben, how much candy was in the original pile? Express the number of pieces in a simplified form in terms of n.

3. If $2(1-x)f(1-x)-xf(x)=3x^3-2$, express $f(x)$ explicitly as a function of x.

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. f(x) = ____________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016
Round 4: Combinatorics

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. A combination lock has a dial with integers from 0 to 10, inclusive. Each lock has a unique three-integer code that opens the lock. If integers cannot be repeated, how many different codes exist for the lock?

2. We say that a set of whole numbers is "good" if we can find three elements in the set such that the difference between any two of the three elements is divisible by 10. If all sets of whole numbers with more than n elements (inclusive) must be "good", determine the minimum value of n.

3. There are five multiple choice questions in a math quiz, and each question has four choices. If Eric has not studied for the quiz and answers all the questions randomly, what is the probability that Eric can receive a grade equal to or above 60%?

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016
Round 5: Analytic Geometry

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. The line $ax+by=1$ has an x-intercept -3 and a y-intercept 4. When $x=3$, what is the value of y?

2. Find the equation of a line passing through the point (2,0) and perpendicular to the line $x+2y=3$.

3. There is an ellipse. The length of its long radius is two and the length of its short radius is one. A straight line goes through the center of the ellipse and forms a 45° angle with a radius. What is/are the possible distance(s) from the intersections of the ellipse and the line to the center of the ellipse?

ANSWERS

(1 pt.) 1. ______________________

(2 pts.) 2. ______________________

(3 pts.) 3. ______________________

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016
Team Round Answer Sheet

1. ________________________

2. ________________________

3. ________________________

4. ________________________ square miles

5. ________________________

6. ________________________

7. ___________ : ___________

8. ________________________ pounds

9. ________________________ miles

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016

Team Round

All answers must either be in simplest exact form or rounded to EXACTLY three decimal places, unless stated otherwise. (3 points each)

APPROVED CALCULATORS ALLOWED

1. In a quadrilateral ABCD, no lines are crossing. ∠CDA = 90°, ∠DAB = 60° and ∠BCD = 120°. If CD = 2 and AD = $\sqrt{3}$, what is the length of BC?

2. Suppose $2x - y + z = 5$, and $x + y + 2z = 7$. If x, y, z are all <u>positive integers</u>, find all possible solutions (x, y, z).

3. If $f(x) = 3x^2 - 9$ and $g(x) = 2x + a$, find the value of a so that the graph of f composed with g, $f \circ g(x)$ or $f(g(x))$, crosses the y-axis at 216.

4. Harris is walking at a speed of 4 miles/hour freely, but he is only allowed to walk towards east, west, north or south. That is, while he can walk in different directions alternatively, but he <u>cannot</u> walk in any directions other than these four. What is the total area of all the places that he could possibly stop at after an hour?

5. Find the total area enclosed by the outermost perimeter of:

$(x-4)^2 + (y-4)^2 = 32$

$(x+4)^2 + (y-4)^2 = 32$

$(x+4)^2 + (y+4)^2 = 32$

$(x-4)^2 + (y+4)^2 = 32$

6. Evaluate the formula:

$$\sum_{a=1}^{\infty}\sum_{b\geq a}^{\infty}\frac{1}{2^a 3^b}$$

7. In triangle ABC, ∠B = 90°. Find a point D on the line segment BC such that ∠DAC = 90° - 2∠BCA. BD = 3. If the length of AB and CD are both positive integers, what is the smallest possible value of CD?

8. Elena is an immortal dragon that lives beside a village and really likes gold and mathematics. Each year, the village gives her two pounds of gold as a reward for her protection. Elena <u>puts all her gold together</u> and calculates the <u>reciprocal</u> of the weight (in pounds) of the gold she has. Then, Elena stores the reciprocal of the amount of gold she

has, and spends the rest of her gold for food. After many many years, how much gold (in pounds) will she eventually store, assuming she does the same thing every year?

9. Rick is looking downward from inside an elevator. Due to the design of the wall of the elevator, Rick can only look downward at an angle between 30° and 60°. Assume the land is flat and the view is clear. If Rick looks around and realizes that he can view 0.24π square miles, how many miles is Rick above ground right now?

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016 Answer Key

Round 1:

1. $\frac{60}{13}$
2. 40 (J. Bryan Sullivan, Hudson)
3. 5 (Worcester Academy)

Round 2:

1. 1 (Shepherd Hill)
2. 48
3. $\frac{40}{3}$ or $13\frac{1}{3}$ or $13.\bar{3}$ (AMSA)

Round 3:

1. 4 (Quaboag)
2. $6n-18$ or $6(n-3)$
3. $f(x)=-x^2+6x-6$ (Algonquin)

Round 4:

1. 990 (Tantasqua)
2. 21 (Inspired by Manhattan Mathematical Olympiad 2005)
3. $\frac{53}{512}$

Round 5:

1. 8 (Bancroft)
2. $y=2x-4$ or $2x-y-4=0$ or $2x-y=4$ or $(y - 0) = 2(x - 2)$ (Shepherd Hill)
3. $\frac{2\sqrt{10}}{5}$

TEAM Round:

1. $\frac{1}{2}$ or 0.5
2. (2, 1, 2), (3, 2, 1) (in either order)
3. $\pm 5\sqrt{3}$ or $5\sqrt{3}$, $-5\sqrt{3}$ (Groton-Dunstable)
4. 32
5. $16\sqrt{2}+64\pi$ (Montachusetts Regional Vocational Technical School)
6. $\frac{3}{10}$ or 0.3 (Inspired by Putnam Exam 2015 B-4)
7. 9
8. $-1+\sqrt{2}$
9. 0.3

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016 - Solutions
Round 1: Similarity and Pythagorean Theorem

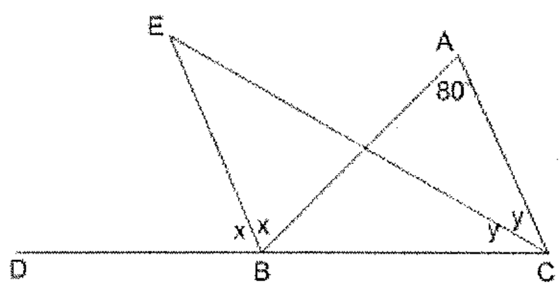
All answers must be in simplest exact form in the answer section

NO CALCULATOR AND RULER ALLOWED

1. In $\triangle$ABC, AB = 5, BC = 12 and AB $\perp$ BC. Find the distance from point B to AC.

Solution: The distance from B to BC is the length of the height on the line AC. Since AB $\perp$ BC, the area of the triangle is $\frac{1}{2}\times 5\times 12 = 30$. By Pythagorean theorem, we obtain that AC = 13, and AC times the height on AC equals twice the area of the triangle. Therefore, the height on AC is 60/13.

2. CD is a line segment, and point B is on line segment CD. CE bisects $\angle$ACD and BE bisects $\angle$ABD. If $\angle$BAC = 80°, find $\angle$BEC in degrees.



Solution: Let $\angle$ABE = $\angle$EBD = x and let $\angle$ACE = $\angle$ECB = y. The graph is shown on the right. Since $\angle$ABD is the supplement of $\angle$ABC, and since $\angle$ABC is an angle of triangle ABC,

$$\angle ABD = \angle A + \angle ACB$$

Substitute in the variables,

$$2x = 80^\circ + 2y$$

Similarly, we can view $\angle$EBD as the supplement of $\angle$EBC in triangle EBC, and derive

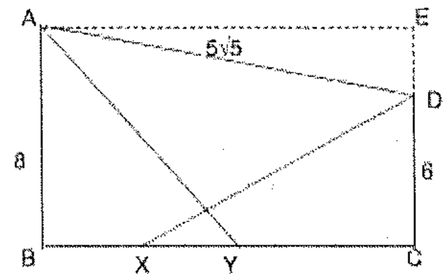
$$\angle EBD = \angle E + \angle ECB$$

Substitute in the variables,

$$x = \angle E + y$$

Since we already know that $2x = 80^\circ + 2y$, divide the equation by 2 on both sides, and we have $x = 40^\circ + y$, so $\angle E = 40^\circ$.

3. $\angle B$ and $\angle C$ are right angles, $AB = 8$, $DC = 6$, $AD = 5\sqrt{5}$, $BX = XY$ and $XD = AY$. Find YC.



Solution: As the graph shows, extend DC and draw a line through A parallel to BC such that the parallel line intersects the extension of CD at E. We thus obtain a rectangle ABCE. First, by rectangle we observe that

$$CE = AB = 8$$

so

$$DE = CE - CD = 8 - 6 = 2$$

Since $\angle E = 90°$, by pythagorean theorem, we can derive that

$$AE^2 = AD^2 - DE^2 = 121$$
$$AE = 11.$$

Let $BX = XY = m$ and $AY = DX = n$, then we can express CX as

$$CX = BC - m = AE - m = 11 - m$$

Note that we have right triangles ABY and CDX, so

$$AB^2 + BY^2 = AY^2$$
$$CX^2 + CD^2 = DX^2$$

Plug in all the variables and numbers, we obtain

$$8^2 + (2m)^2 = y^2$$
$$(11 - m)^2 + 6^2 = y^2$$

Equate the two equations,

$$8^2 + (2m)^2 = y^2 = (11 - m)^2 + 6^2$$
$$3m^2 + 22m - 93 = 0$$

To solve the equation, note that

$$3m^2 + 22m - 93 = (3m + 31)(m - 3)$$

So $m = 3$ is the only positive solution for the equation. Therefore, $m = 3$, and

$$CY = CX - XY = (11 - m) - m = 5$$

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016 - Solutions
Round 2: Algebra 1

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If $2x+y=5$ and $y+2z=3$, what is the value of $x-z$?

Solution: Subtract the second equation from the first equation, get

$$(2x+y)-(y+2z)=5-3$$
$$2x+y-y-2z=2$$
$$2x-2z=2$$
$$x-z=1$$

2. Philip is driving back home. For the first half of the trip ("half" in terms of distance), he drives at 40 miles per hour. For the second half of the trip, he drives at 60 miles per hour instead. What is the average speed of this trip?

Solution 1: Average speed is equal to total distance divided by total travel time. Assume the total distance is d. The time spent on the first half of the trip is $\frac{d}{2}\div 40=\frac{d}{80}$, and the time spent on the second half of the trip is $\frac{d}{2}\div 60=\frac{d}{120}$. Therefore, the total time spent on the trip is $\frac{d}{80}+\frac{d}{120}=\frac{d}{48}$, and the average speed throughout the trip is $d\div\frac{d}{48}=48$ miles per hour.

Solution 2: Assume the total distance is 240 miles. We choose this number because half of 240 is divisible by 40 and 60 and it is easy to compute.

For the first half of the trip, the total distance is 120 miles, and the total travel time is thus $120\div 40=3$ hours. For the second half of the trip, the total travel time is $120\div 60=2$ hours. The total travel time for the whole trip is thus 5 hours, and the average speed is $240\div 5=48$ miles per hour.

3. Rachel walks on a moving walkway at Logan airport. She walks at a constant speed all the time, both back-and-forth. It takes her 15 seconds to walk to the end of the walkway and then back to the starting point. If the walkway doubles its speed, it takes her 24 seconds to walk to the end and back to the starting point. If the moving walkway is off, how long will it take for Rachel to walk to the end of the walkway and back to the starting point at her constant speed?

Solution: Let the speed of the walkway be w and the speed of Rachel be r, and assume the length of the walkway (that is, the distance) is l. When Rachel walks on the moving walkway, she must walk in the same direction as the walkway once and walk against the direction once.

When Rachel and the walkway go in the same direction, Rachel is moving at the speed of $(w+r)$, so the time cost is $\frac{d}{w+r}$. When Rachel and the walkway go in opposite directions, the time cost is $\frac{d}{w-r}$. From the question, the total time cost is

$$\frac{d}{w-r}+\frac{d}{w+r}=15$$

Then, as the walkway doubles its speed, the speed of the walkway becomes 2r, and the total time cost becomes

$$\frac{d}{w-2r}+\frac{d}{w+2r}=24$$

If we expand the two equations, we have

$$\begin{cases} l(r+w)+l(r-w)=15(r-w)(r+w) \\ l(r+2w)+l(r-2w)=24(r-2w)(r+2w) \end{cases}$$

$$\begin{cases} lr+lw+lr-lw=15(r^2-w^2) \\ lr+2lw+lr-2lw=24(r^2-4w^2) \end{cases}$$

$$\begin{cases} 2lr=15(r^2-w^2) \\ 2lr=24(r^2-4w^2) \end{cases}$$

Equate the two formulas and expand, and we obtain

$$15r^2-15w^2=2lr=24r^2-96w^2$$
$$81w^2=9r^2$$
$$9w^2=r^2$$
$$3w=r$$

Plug the result into the first equation we obtain, we have

$$15=\frac{l}{r-w}+\frac{l}{r+w}=\frac{l}{r-\frac{r}{3}}+\frac{l}{r+\frac{r}{3}}=\frac{3}{2}\times\frac{l}{r}+\frac{3}{4}\times\frac{l}{r}=\frac{9}{4}\times\frac{l}{r}$$

Therefore, if Rachel walks without the help of the walkway, and if she is walking down the walkway and back, the total time is

$$\frac{2l}{r}=2\times(15\times\frac{4}{9})=\frac{40}{3}\text{ seconds}$$

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016 - Solutions
Round 3: Functions

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. If c(x) = x mod 12 is a clock function (for example, c(0) = 12, c(13) = 1 and c(-8) = 4), find the value of c(c(42) + c(-14)).

Solution: Since it is the same time to rotate a clock by 12 hours,

$$c(42) = c(42 - 12) = c(30)$$
$$c(30) = c(18) = c(6) = 6$$
$$c(-14) = c(-2) = 10$$

Therefore,

$$c(\ c(42) + c(-14)\ ) = c(6 + 10) = c(16) = c(4) = 4$$

2. Abby, Ben and Catherine are collecting candies at Halloween. Afterwards, Abby puts all candies into a pile and gives half plus three pieces of the total candy to Ben. Then, Ben gives a third plus two pieces of his candy to Catherine. If Catherine receives n (Assume $n>3$) pieces of candy from Ben, how much candy was in the original pile? Express the number of pieces in a simplified form in terms of n.

Solution 1: Assume Abby receives x pieces of candies, then Ben receives $\frac{x}{2}+3$ pieces of candies, and Catherine receives

$$(\frac{x}{2}+3)\times\frac{1}{3}+2=\frac{x}{6}+3 \text{ pieces}$$

Now, if we assume Catherine receives n pieces of candies and want to express x in terms of n, let

$$n=\frac{x}{6}+3$$

and we have

$$6n=x+3\times6$$
$$x=6n-18$$

Solution 2: We can also plug in several values for n and observe the pattern.
If Catherine has n = 4 pieces of candies, Ben has 6 pieces of candies, and Abby has 6.
If Catherine has n = 5 pieces of candies, Ben has 9 pieces of candies, and Abby has 12.
If Catherine has n = 6 pieces of candies, Ben has 12 pieces of candies, and Abby has 18.
If Catherine has n = 7 pieces of candies, Ben has 15 pieces of candies, and Abby has 24.
We observe the pattern that Abby receives 6 more pieces of candies when n increases by 1. One could guess that $x = 6n - 18$ is the simplest relation that satisfies the pattern.

3. If $2(1-x)f(1-x) - xf(x) = 3x^3 - 2$, express $f(x)$ explicitly as a function of x.

Solution: We can treat the expression as an equation with two variables. One variable is $f(x)$, and the other one is $f(1-x)$. In the original equation, if we substitute x with $(1-x)$, $f(x)$ becomes $f(1-x)$, and $f(1-x)$ becomes $f(1-(1-x)) = f(x)$, so we obtain a new equation with the same two variables, and we have two equations in total to solve for two variables:

$$\begin{cases} 2(1-x)f(1-x) - xf(x) = 3x^3 - 2 \\ 2xf(x) - (1-x)f(1-x) = 3(1-x)^3 - 2 \end{cases}$$

If we multiply the second equation by 2 and add the first equation to the product, we can cancel out the $(1-x)f(1-x)$ part and obtain the expression for $xf(x)$. The sum of the left hand side becomes

$$\begin{aligned} &(2\times 2xf(x) - 2\times(1-x)f(1-x)) + (2(1-x)f(1-x) - xf(x)) \\ &= 4xf(x) - 2(1-x)f(1-x) + 2(1-x)f(1-x) - xf(x) \\ &= 3xf(x) \end{aligned}$$

and the right hand side becomes

$$\begin{aligned} &2\times(3(1-x)^3 - 2) + (3x^3 - 2) \\ &= 2\times(3\times(-x^3 + 3x^2 - 3x + 1) - 2) + (3x^3 - 2) \\ &= 2(-3x^3 + 9x^2 - 9x + 1) + 3x^3 - 2 \\ &= -3x^3 + 18x^2 - 18x \end{aligned}$$

Therefore,

$$3xf(x) = -3x^3 + 18x^2 - 18x$$

$$f(x) = -x^2 + 6x - 6$$

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016 - Solutions
Round 4: Combinatorics

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. A combination lock has a dial with integers from 0 to 10, inclusive. Each lock has a unique three-integer code that opens the lock. If integers cannot be repeated, how many different codes exist for the lock?

Solution: Assume we are picking the integers one by one. The first integer can take any of the 11 values, the second integer can take any of the 10 rest values, and the third integer can take any of the 9 rest values. Therefore, the total number of combinations is $11 \cdot 10 \cdot 9 = 990$.

2. We say that a set of whole numbers is "good" if we can find three elements in the set such that the difference between any two of the three elements is divisible by 10. If all sets of whole numbers with more than n elements (inclusive) must be "good", determine the minimum value of n.

Solution 1: This is an application of the pigeonhole principle. Look at each number mod 10. If two numbers are equal in mod 10, then their difference is a multiple of 10. There are only 10 possible such outcomes in mod 10, so the minimum number in the Set is 21, for it to be guaranteed to be "good".

Solution 2: We can first find out the maximum of n such that the sets with n elements could be "not good", and then add one more number to exceed the maximum so that all sets with $(n + 1)$ elements have to be good.

If the difference of two numbers is divisible by 10, the last digits of the two numbers must be the same. Therefore, for each possible last digit, we can only assign 2 numbers. If we have 3 numbers of the same last digit, their differences must be divisible by 10. If each last digit correspond to only 2 numbers, there is no way to find the 3rd number such that it shares the same last digit with the previous four numbers.

Since there can be 10 different last digits, and each last digits can only be assigned 2 numbers, we can construct up to 20 numbers such that we cannot choose 3 numbers whose differences are divisible by 10. Then, if we add one more number to the set, the number must have *some* last digit, so we must have a set of 3 numbers with the same last digit in this case. Since this is already the "worst" scenario, sets with 21 or more elements must be "good" in this sense. Therefore, n = 21.

3. There are five multiple choice questions in a math quiz, and each question has four choices. If Eric has not studied for the quiz and answers all the questions randomly, what is the probability that Eric can receive a grade equal to or above 60%?

Solution 1: If Eric is answering randomly and gets above 60%, he could get 3, 4 or 5 questions correct and 2, 1 or 0 questions wrong. Since these cases are exclusive, we can calculate them one by one and add them together.

1. If Eric gets 3 questions right and 2 questions wrong, there are ${}_2C_5 = 10$ ways of assigning the two wrong questions. Since there is 1 choice to get a question right and 3 choices to get it wrong, there are $1^3 \times 3^2 = 9$ ways in total to have 3 specific questions right and 2 specific questions wrong. In total, there are $10 \times 9 = 90$ ways to get 3 questions right and 2 questions wrong.

2. If Eric gets 4 questions right and 1 question wrong, there are 5 ways of assigning the wrong question, and there are $1^4 \times 3 = 3$ ways of answering questions to have one and only one specific question wrong. There are 15 combinations in total.

3. If Eric gets 5 questions right and 0 question wrong, he has to choose all correct choices and there is only 1 way of doing that.

In total, there are $90 + 15 + 1 = 106$ cases that Eric could get 60% or higher. The total number of all possible cases is $4^5 = 1024$ because there are four ways of answering each question. The probability is the number of cases that Phillip gets 60% or above divided by the total number of all cases, which is $106 \div 1024 = \dfrac{53}{512}$.

Solution 2: An alternative way is to use Pascal's triangle to directly determine the probability in each case. The result is that

$$
\begin{aligned}
& 1 \times (\frac{1}{4})^5 \times (\frac{3}{4})^0 + 5 \times (\frac{1}{4})^4 \times (\frac{3}{4})^1 + 10 \times (\frac{1}{4})^3 \times (\frac{3}{4})^2 \\
&= \frac{1}{1024} + \frac{15}{1024} + \frac{90}{1024} \\
&= \frac{106}{1024} \\
&= \frac{53}{512}
\end{aligned}
$$

WORCESTER COUNTY MATHEMATICS LEAGUE

Varsity Meet 3 - January 27, 2016 - Solutions
Round 5: Analytic Geometry

All answers must be in simplest exact form in the answer section

NO CALCULATOR ALLOWED

1. The line $ax+by=1$ has an x-intercept -3 and a y-intercept 4. When $x=3$, what is the value of y?

Solution 1: The x-intercept of a line is the intersection of the line with x-axis. When the line intersects x-axis, y value must be zero on the x-axis. Since $x=-3$ at the intersection,

$$a\times(-3)+b\times 0=1$$
$$a=-\frac{1}{3}$$

For the same reason, the y-intercept is the intersection on the y-axis, so $x=0$ at this point. Since $y=4$,

$$a\times 0+b\times 4=1$$
$$b=\frac{1}{4}$$

Therefore, the formula of the line is

$$-\frac{1}{3}x+\frac{1}{4}y=1$$

When $x=3$,

$$-\frac{1}{3}\times 3+\frac{1}{4}y=1$$
$$y=4\times(1+\frac{1}{3}\times 3)=8$$

Solution 2: Since the x-intercept is -3, the point (-3, 0) is on the line. Since the y-intercept is 4, the point (0, 4) is on the line. Therefore, as we progress from x = -3 to x = 0, the value of y increases by 4. If we progress from x = 0 to x = 4, the value of y should increase by 4 again because it is a straight line, and thus giving us y = 8.

Solution 3: Given (-3,0) and (0, 4) are on the line. The equation of the line is

$$y=\frac{4}{3}(x-(-3))$$

Equivalently,

$$y=\frac{4}{3}x+4$$

So when x = 3, y= 4 + 4 = 8.

2. Find the equation of a line passing through the point (2, 0) and perpendicular to the line $x+2y=3$.

Solution: Assume the line line passing through (2, 0) takes the form

$$y_1=kx_1+b$$

We can write the other line as

$$x_2+2y_2=3$$

which can be written in the form

$$y_2=-\frac{1}{2}x+\frac{3}{2}$$

If the first line is perpendicular to the second line, we know that the product of the slopes of two perpendicular lines is equal to -1, so $k=2$ and

$$y_1=2x_1+b$$

Since the line passes through (2, 0), plug in the point and we obtain

$$0=2\times2+b$$
$$b=-4$$

and the equation we are looking for is $y=2x-4$.

3. There is an ellipse. The length of its long radius is two and the length of its short radius is one. A straight line goes through the center of the ellipse and forms a 45° angle with an radius. What is/are the possible distance(s) from the intersections of the ellipse and the line to the center of the ellipse?

Solution: We first need to establish a coordinate system. As the graph shows, let the center of the ellipse be the origin of the coordinate system, point O, and let the direction of the long radius be the positive direction of the x-axis. One 45-degree line is indicated in the graph, and we know that if a line has a 45 degree angle with respect to the x-axis, it can have the formula $y=x$ or $y=-x$. Also, one can check that the same two lines are the two lines that form 45° angles with the short radius. For simplicity, we take $y=x$. Let $y=x$ intersect the ellipse at point A.

As we have the length of the two radiuses of the ellipse, we know that the formula for the ellipse is

$$\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$$

Since the 45-degree line can be represented as $y=x$, we just need to solve for a system of equations containing the two formulas to find the intersection of the line and the ellipse. Substitute $y=x$ into the equation of the ellipse, we get

$$\frac{x^2}{4}+x^2=1$$

$$x=\frac{2}{\sqrt{5}}$$

Note that A is in the first quadrant, so both x and y values are positive. Then

$$y=x=\frac{2}{\sqrt{5}}$$

By the Pythagorean Theorem, the distance from point A to point O is

$$AO=\sqrt{(\frac{2}{\sqrt{5}}-0)^2+(\frac{2}{\sqrt{5}}-0)^2}$$

$$=\sqrt{\frac{4}{5}+\frac{4}{5}}$$

$$=\sqrt{\frac{8}{5}}=\frac{\sqrt{8}}{\sqrt{5}}=\frac{\sqrt{8}\times\sqrt{5}}{\sqrt{5}\times\sqrt{5}}$$

$$=\frac{\sqrt{40}}{5}=\frac{2\sqrt{10}}{5}$$

WORCESTER COUNTY MATHEMATICS LEAGUE

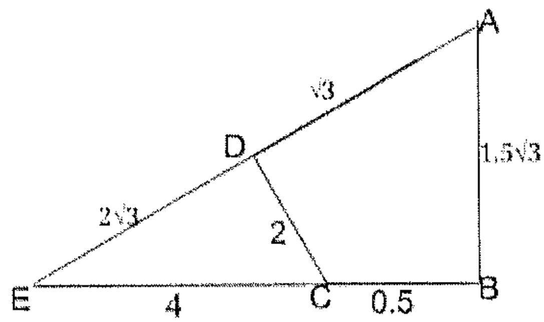
Varsity Meet 3 - January 27, 2016 - Solutions

Team Round

All answers must be in simplest exact form, unless stated otherwise. (3 points each)

APPROVED CALCULATORS ALLOWED

1. In a quadrilateral ABCD, no lines are crossing. ∠CDA = 90°, ∠DAB = 60° and ∠BCD = 120°. If CD = 2 and AD = $\sqrt{3}$, what is the length of BC?



Solution 1: As the graph shows, extend line AD and BC so that the two lines intersect at point E.

Since the four angles of a quadrilateral adds up to 360 degrees, we know that ∠B = 90°. Therefore, triangle AEB is a right triangle. Since $CD \perp AE$, we know that triangle CDE is also a right triangle. Since ∠BCD = 120°, ∠DCE = 60°, and triangle CDE is a special right triangle with ∠E = 30°.

For such special right triangle, the ratio of the length of the three edges is $1{:}\sqrt{3}{:}2$. Thus

$$1{:}\sqrt{3}{:}2 = DC{:}DE{:}CE = 2{:}DE{:}CE$$
$$DE = 2\sqrt{3}$$

Since $AD = \sqrt{3}$,

$$AE = AD + DE = 3\sqrt{3}$$

Since ∠DAB = 60°, this is again the special right triangle, and we can calculate that

$$AB = 1.5\sqrt{3}$$
$$BE = 4.5$$
$$BC = BE - CE = 0.5.$$

Solution 2: We can also connect AC and use Pythagorean theorem to derive that $AC = \sqrt{7}$. Then, similar to the previous solution, we can calculate the length of AB, and apply Pythagorean theorem to triangle ABC to calculate the length of BC and get the same result.

By Pythagorean theorem,

$$AC^2 = AD^2 + CD^2 = 3 + 4 = 7$$
$$AC = \sqrt{7}$$

Similar to solution 1, we derive

$$AB = 1.5\sqrt{3}$$

So

$$BC = \sqrt{AC^2 - AB^2} = \sqrt{(\sqrt{7})^2 - (1.5\sqrt{3})^2} = \sqrt{7 - \frac{27}{4}} = \sqrt{\frac{1}{4}} = \frac{1}{2}$$

2. Suppose $2x - y + z = 5$ and $x + y + 2z = 7$. If x, y, z are all positive integers, find all possible solutions (x, y, z).

Solution: we need to simplify the two equations to narrow down the range of the variables. First, we can add the two equations together to cancel out y:

$$(2x - y + z) + (x + y + 2z) = 5 + 7$$
$$3x + 3z = 12$$
$$x + z = 4$$

Since $x + z = 4$ and all variables are positive integers, x could only take 1, 2 or 3, or z will not be positive integer. Therefore, there are only three cases to consider, x = 1, 2 or 3, and by $x + z = 4$, z would be 3, 2 and 1 respectively.

To calculate the value of y in the three cases, plug the values of x and z for each case into the equation $2x - y + z = 5$. We obtain

If $x = 1$ and $z = 3$, we obtain

$$2 \times 1 - y + 3 = 5$$
$$y = 0$$

In this case y is not a positive integer.

If $x = 2$ and $z = 2$,

$$2 \times 2 - y + 2 = 5$$
$$y = 1$$

The solution is thus (2, 1, 2)

If $x=3$ and $z=1$,

$$2\times3-y+1=5$$
$$y=2$$

The solution is (3, 2, 1).

So the only two sets of solutions are (2, 1, 2) and (3, 2, 1).

3. If $f(x)=3x^2-9$ and $g(x)=2x+a$, find the value of a so that the graph of *f* composed with *g*, $f\circ g(x)$ or $f(g(x))$, crosses the y-axis at 216.

Solution: put the two functions together and simplify,

$$f\circ g(x)=3(2x+a)^2-9$$
$$=3(4x^2+4ax+a^2)-9$$
$$=12x^2+12ax+3a^2-9$$

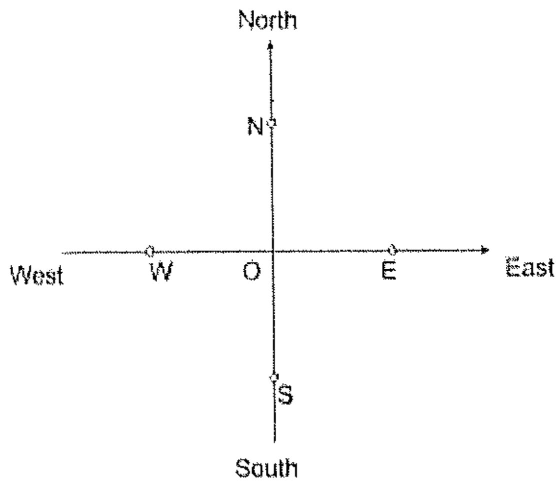
When the function crosses the y-axis, the value of *x* is zero. Plug in *x* = 0 and let *y* = 216, we obtain

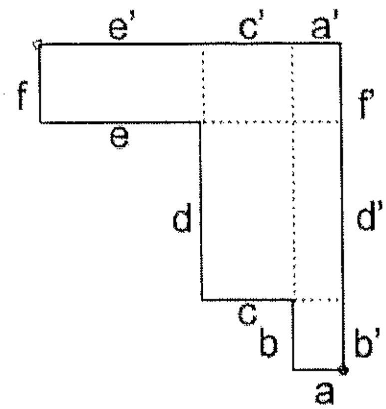
$$216=f\circ g(0)=12\times0^2+12a\times0+3a^2-9=3a^2-9$$
$$3a^2=216+9=225$$
$$a^2=75$$
$$a=\pm5\sqrt{3}$$

4. Harris is walking at a speed of 4 miles/hour freely, but he is only allowed to walk towards east, west, north or south. That is, while he can walk in different directions alternatively, he cannot walk in any directions other than these four. What is the total area of all the places that he could possibly stop at after an hour?

Solution: we can first find out the boundary of Harris's movements, and Harris can stay in any point inside the boundary. Therefore, the area inside the boundary is the area of all the places he could stop.

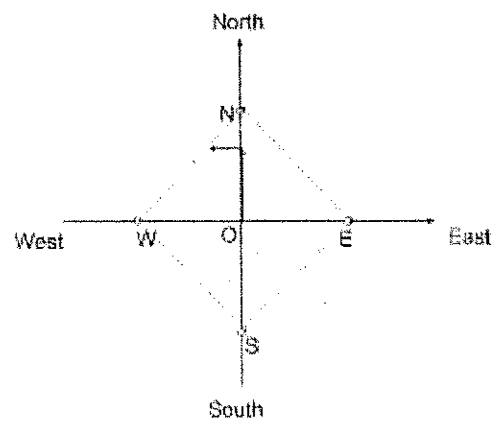
First, if Harris walks straight in any one of the four directions, he can reach 4 miles east, west, north or south the furthest, so these four points must be in the boundary, which are shown below on the left:





Then, we want to show that, if Harris walks in two directions alternatively, let's say north and west, we can always substitute a complicated walk with a simplier one. As the graph above on the right shows, if a random northwestern path goes in the order of "a - b - c - d - e - f", we can rearrange all these paths to make it a nice "b' - d' - f' - a' - c' - e'" path. Therefore, when we consider the furthest he could get when he walks in only two directions, we only need to consider the case that he walks in one direction once and another direction once.

We still study the combination of north and west. Since he walks for an hour in the same speed, the length of the path is the same. We can tentatively draw some paths, like the graph on the right shows. Notice that the endpoints of all the paths actually fall on the line segment NW! Therefore, if Harris only walks towards north and west, the boundary of his walk is bounded by the line segment NW. Since he can also walk towards north and east, south and west, and south and east, the boundary of his walk is the square NESW.

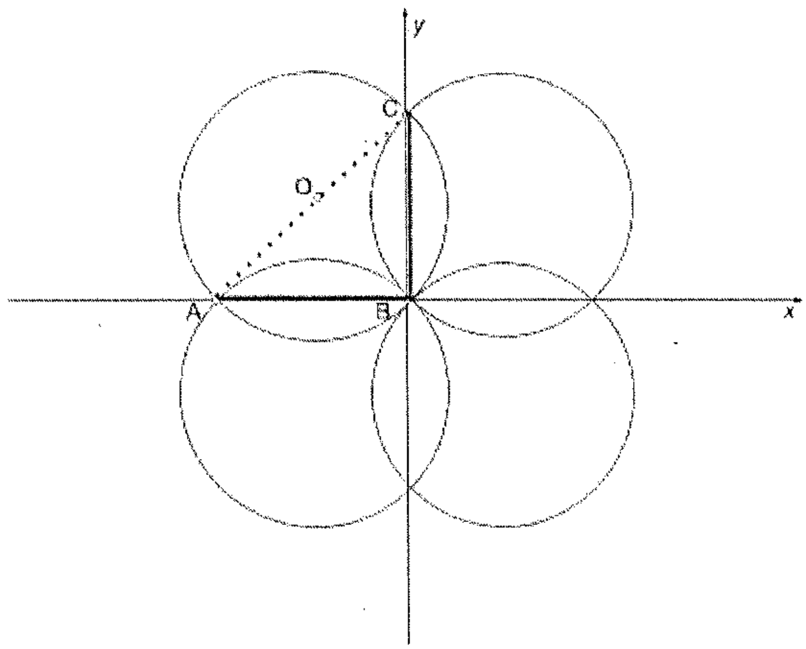


Another way of viewing this problem is that, if Harris walks towards the north for a while and then towards the west all the way to the end, it is like to fold the line ON towards the left, and intuitively we can see that the end point falls on the line segment NW.

If Harris walks in a combination of three directions, two of the directions must cancel out with each other, so it is less efficient and the boundary for three directions must not exceed the boundary for two directions. Therefore, the square NESW is the boundary for Harris's walk. Since ON = 4, the area of triangle ONW is 8 square miles, and the area of the square is thus 32 square miles. As we explained before, Harris could stop in any of the points inside the square, so the total area of the places he could end up with is 32 square miles.

5. Find the total area enclosed by the outermost perimeter of:

$$(x-4)^2+(y-4)^2=32$$
$$(x+4)^2+(y-4)^2=32$$
$$(x+4)^2+(y+4)^2=32$$
$$(x-4)^2+(y+4)^2=32$$



Solution: The four circles are shown in the graph. The area enclosed by line segment AB, BC and arc CA is a quarter of the total area we are going to calculate. We can thus calculate the quarter and multiply that by 4 to derive the total area. To calculate the quarter, we can break the area into triangle ABC and the area between line AC and arc AC.

To calculate the area of triangle ABC, we need to know the length of BC. Notice that C is the intersection of the circle with the y-axis, and B is the origin. Therefore, we can calculate the coordinate of C to find out the length of BC. Since circle O is in the second quadrant, the *x* value of point O is negative and the *y* value is positive, so the equation is $(x+4)^2+(y-4)^2=32$. At point C, the *x* coordinate is zero. Since point C is on the circle O, we can plug in the x-coordinate of C, and get

$$(0+4)^2+(y-4)^2=32$$
$$(y-4)^2=32-4^2=16$$
$$y=4+4=8 \text{ OR } y=-4+4=0$$

Since point C is on the positive half of the y-axis, its *y* value must be positive, and thus the *y* value is 8, which is also the length of BC. Another way to find out the length of BC is to notice that triangle BOC is an isosceles right triangle. Since the graph is symmetric, the length of AB is also 8. Therefore, the area of triangle ABC is 32.

To calculate the area between the arc AC and the line segment AC, first we can calculate that the length of AC is $8\sqrt{2}$. From the formula of the circles, we know that the length of radius is $4\sqrt{2}$, so AC is the diameter of the circle, and thus the area enclosed is half of the circle. One can calculate that a half of the circle is 16π. Therefore, a quarter of the graph is $4\sqrt{2}+16\pi$, and the whole thing is $16\sqrt{2}+64\pi$.

6. Evaluate the formula:

$$\sum_{a=1}^{\infty}\sum_{b\geq a}^{\infty}\frac{1}{2^a3^b}$$

Solution: To evaluate the formula, we first need to fix a to be a positive integer. Then, the formula becomes

$$\sum_{b\geq a}^{\infty}\frac{1}{2^a3^b}=\frac{1}{2^a}\times\sum_{b\geq a}^{\infty}\frac{1}{3^b}=\frac{1}{2^a3^a}\times\sum_{b=0}^{\infty}\frac{1}{3^b}$$

By the formula about the sum of infinite geometric sequence, we obtain that

$$\sum_{b=0}^{\infty}\frac{1}{3^b}=\frac{1}{1-\frac{1}{3}}=\frac{3}{2}$$

Therefore, the original formula becomes

$$\sum_{a=1}^{\infty}\sum_{b\geq a}^{\infty}\frac{1}{2^a3^b}=\sum_{a=1}^{\infty}\frac{1}{2^a3^a}\times\sum_{b=0}^{\infty}\frac{1}{3^b}=\sum_{a=1}^{\infty}\frac{1}{6^a}\times\frac{3}{2}$$

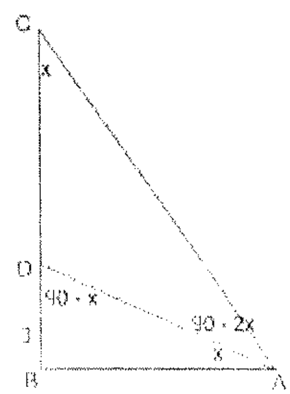
By the formula about the sum of infinite geometric sequence,

$$\sum_{a=1}^{\infty}\frac{1}{6^a}=\frac{1}{6}\times\frac{1}{1-\frac{1}{6}}=\frac{1}{5}$$

And the value of the original formula is

$$\sum_{a=1}^{\infty}\frac{1}{6^a}\times\frac{3}{2}=\frac{1}{5}\times\frac{3}{2}=\frac{3}{10}$$

7. In triangle ABC, ∠B = 90°. Find a point D on the line segment BC such that ∠DAC = 90° - 2∠BCA. BD = 3. If the length of AB and CD are both positive integers, what is the smallest possible value of CD?



Solution: as the graph shows, let ∠C = x and thus ∠DAC = 90° - 2x. Then, ∠BDA = ∠C + ∠CAD = 90° + x because ∠BDA is the supplement of ∠ADC, which is an angle in the triangle ACD. Since ∠B = 90°, ∠BAD = 90° - ∠ADB = x.

We notice that ∠BAD = ∠C. Since triangle ABC and triangle ABD both share angle B, and since ∠BAD = ∠C, we can derive that triangle ABD is similar to triangle CBA. Therefore, DB : AB = AB : CB. Notice that the first AB is in triangle ABC, and the second AB is in triangle ABD. This expression is equivalent to DB • CB = AB^2.

Since the length of CD and DB is integers, the length of BC is also an integer. Since the length of AB is an integer too, DB • CB must be a perfect square. The smallest perfect square containing 3 is $3^2 = 9$, but that means CB = 9 ÷ DB = 3, but BC has to be longer than BD so that CD can have a positive length. The second smallest perfect square is $(3\times 2)^2 = 36$, and in this case CB = 36 ÷ DB = 12. AB = 6 and CD = CB - BD = 12 - 3 = 9.

8. Elena is an immortal dragon that lives beside a village and really likes gold and mathematics. Each year, the village gives her two pounds of gold as a reward for her protection. Elena <u>puts all her gold together</u> and calculates the <u>reciprocal</u> of the weight (in pounds) of the gold she has. Then, Elena stores the reciprocal of the amount of gold she has, and spends the rest of her gold for food. After many many years, how much gold (in pounds) will she eventually store, assuming she does the same thing every year?

Solution 1: we can first literally write out the expression of the gold Elena has, and then evaluate the expression. In year one, she receives 2 pounds of gold and that is all she has, so in year 1 she keeps the reciprocal of 2 pounds, which is

$$\frac{1}{2}$$

In year 2 she receives another 2 pounds of gold, and she puts the gold together with the half left from last year. Then, she keeps the reciprocal of the sum, which is

$$\frac{1}{2+\frac{1}{2}}$$

The process repeats in year 3:

$$\frac{1}{2+\frac{1}{2+\frac{1}{2}}}$$

Year 5:

$$\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2+\frac{1}{2}}}}}$$

After many many years, we can follow the rule and see that Elena's gold is going to be:

$$\frac{1}{2+\frac{1}{2+\frac{1}{\cdots}}}$$

To evaluate the expression, the classical method is to assume that the expression is equal to x. That is, assume

$$x = \cfrac{1}{2+\cfrac{1}{2+\frac{1}{x}}}$$

Then, we have

$$x = \cfrac{1}{2+\cfrac{1}{2+\frac{1}{x}}} = \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\dots}}} = \frac{1}{2+x}$$

Multiply both terms with *x* by the denominator, we get

$$x(2+x)=1$$
$$x^2+2x-1=0$$

Use the formula for the solution of quadratic equations, we get

$$x = \frac{-b\pm\sqrt{b^2-4ac}}{2a} = \frac{-2\pm\sqrt{(-2)^2-4(-1)}}{2} = \frac{-2\pm\sqrt{8}}{2} = -1\pm\sqrt{2}$$

Since Elena cannot have a negative amount of gold, she should finally have $-1+\sqrt{2}$ pounds of gold after many many years.

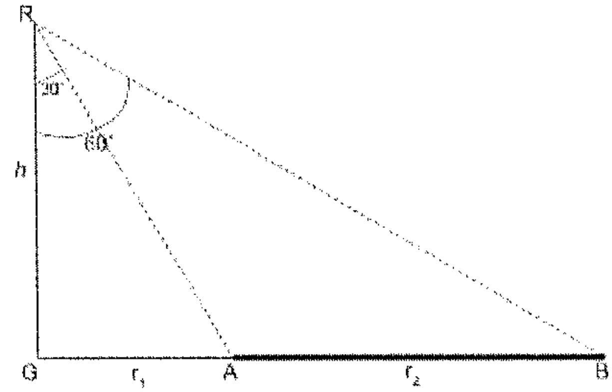
Solution 2: note that in the whole process, when Elena does not have a lot of gold, the reciprocal is large and she keeps much gold. When Elena has a lot of gold, the reciprocal is small and she spends more gold. Therefore, we could guess that after many many years, the amount of Elena's gold should approach a certain number and reach an equilibrium.

Assume that number to be *x*. Since this is the number that the weight of Elena's gold approaches, the amount of gold should not change after a year. Therefore, we derive

$$\frac{1}{2+x} = x$$

We can then follow the same arithmetic process in solution 1 and derive the answer.

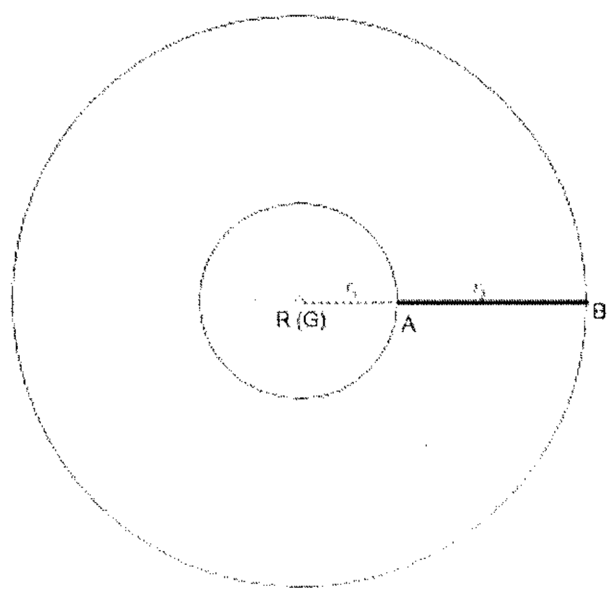
9. Rick is looking downward from inside an elevator. Due to the design of the wall of the elevator, Rick can only look downward at an angle between 30° and 60°. Assume the land is flat and the view is clear. If Rick looks around and realizes that he can view 0.24π square miles, how many miles is Rick above ground right now?



Solution: since the whole scenario is in 3-D, we have to consider the vertical plane and the horizontal plane independently. The vertical graph is shown on the right. The graph is like what a person could see if she stands on the ground far away from Rick, or we can visualize the graph as a vertical slice of the situation.

Point R is Rick's (or the elevator's) location. Point G is the point right below Rick. The two dashed lines are the boundary of his vision. The bolded line AB represents the region of the ground that Rick could see. Denote AG = r_1 and AB = r_2. By trigonometry or the property of special right triangle, we can derive that $r_1 = \frac{h}{\sqrt{3}}$ and BG = $h\sqrt{3}$.

Then, we consider the horizontal plane. The graph is shown on the right. The graph is like what a person could see if she is right above Rick on the sky.



Note that, horizontally, the location of Rick and point G overlaps with each other. Rick's vision is actually a circle because he can turn around in 360°. r_1 and r_2 in this graph are the same things as in the previous graph, but they are observed from the sky instead of from the ground.

Since in the r_1 direction, Rick can only see the things inside line AB, and since Rick can turn around 360°, his vision is the ring that is swept by the line segment AB if we rotate AB around point R (or G). The area of the ring is thus

$$A_{\text{big circle}} - A_{\text{small circle}}$$
$$= \pi \cdot BR^2 - \pi \cdot AR^2 = \pi \cdot BR^2 - \pi \cdot r_1^2$$

$$= \pi \cdot (h\sqrt{3})^2 - \pi \cdot (\frac{h}{\sqrt{3}})^2 = \pi 3h^2 - \pi \frac{h^2}{3} = \frac{8\pi h^2}{3}$$

As the question says, Rick's vision actually covers 0.24π square miles, so we have

$$0.24\pi = \frac{8\pi h^2}{3}$$

$$h^2 = 0.24 \times \frac{3}{8} = 0.09$$

$$h = 0.3$$

So Rick is 0.3 miles above ground right now.